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ABSTRACT

In the last decade, a modeling technique has been developed to handle complex input/output analyses where outputs involve joint products and there are no known mathematical relationships linking the outputs or inputs. The technique uses the geometrical concept of a six-dimensional shape called a polytope to analyze the efficiency of each constituent unit--department or university--within a higher education system. Farrell's method represents each institution as a point with multiple input and multiple output (joint products) coordinates. The institutions are plotted in six dimensional space, then joined by line segments to enclose the smallest possible convex region as a six dimensional polytope. If the surface of the polytope is the frontier productivity function, then the distance to the surface from each point measures the production efficiency of that point. The model and variations are largely explained by use of simple examples in two dimensions. Computer algorithms developed to handle the more realistic cases use higher dimensional polytopes. References to more detailed expositions, accounts of actual applications, and existing computer programs enlarge on this description of the methodology. (Author/CM)

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USING A POLYTOPE TO ESTIMATE EFFICIENT PRODUCTION FUNCTIONS OF JOINT PRODUCT PROCESSES

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Association for Institutional Research
Denver, Colorado
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Abstract

The purpose of this paper is to expose the audience to a relatively obscure but promising modeling technique that, in the last decade, has been developed to handle complex input/output analyses where the outputs involve joint products and there are no known mathematical relationships linking the outputs or inputs. This technique uses the geometrical concept of a polytope to analyze the efficiency of each constituent unit—department or university—within a higher education system. The models will be largely explained by use of simple examples in two dimensions. Computer algorithms have been developed to handle the more realistic cases using higher dimensional polytopes. References to more detailed expositions, accounts of actual applications, and existing computer programs will be given.

TM 830 140

USING A POLYTOPE TO ESTIMATE EFFICIENT PRODUCTION FUNCTIONS OF JOINT PRODUCT PROCESSES

Introduction

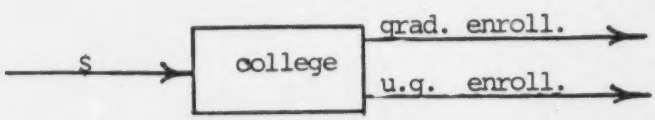
The purpose of this paper is to introduce the interested reader to a little known technique for dealing with the complex and important problem of measuring the efficiency of joint product processes. Surprisingly, this technique is not new—having been slowly evolving for the past 25 years. But, because it is fairly technical, requires the development of computer programs using a sophisticated algorithm, and is very much different from the conventional approaches to this problem, there is a considerable likelihood that it will slip back into oblivion unless some researchers begin to make use of it. Researchers located within large state systems of higher education or multi-campus universities have both the ideal base and the incentive for this type of work, but even those at single campus institutions can use the polytope to assess large groups of departments.

After a quick survey of the economic terms and theory needed, the reader is taken through the research done in this area and shown the developments that have occurred and the nature of the results that have been produced. We conclude by providing references for further reading and the location of available computer programs.

Economic Theory

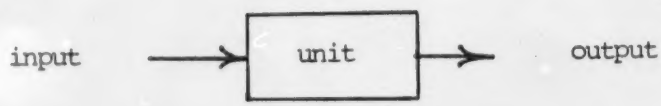
Certainly one of the important problems in higher education is measuring the productive efficiency of its institutions. Many efforts have been made to solve this problem but, although successful gains have been made in measuring some of the outputs and inputs, they all fail to combine these measurements into some useful measure of total efficiency. The difficulty lies in the

complexities attending joint product processes. For example, consider the following:



Clearly the ratios $/Grad and $/UG overstate the cost per output. Even if the input cost can be split into two components $\$ \longrightarrow \$_1+\$_2$ the ratios $\$_1$/Grad and $\$_2$/UG may still be too high. This is due to the joint product phenomenon which results in the production of two outputs (in this case, graduate and undergraduate students) at lower costs than producing them separately. Such processes are not unique to education; most large industrial and agricultural systems are also processes that result in many joint outputs. Thus it is not remarkable that the genesis of the polytope technique began not with educational researchers but rather agricultural economists.

A discussion of "efficiency" usually implies a desire to produce the largest possible amount of output from a given amount of necessary inputs or resources. Taking the simplest possible situation of one input and one output, represented by

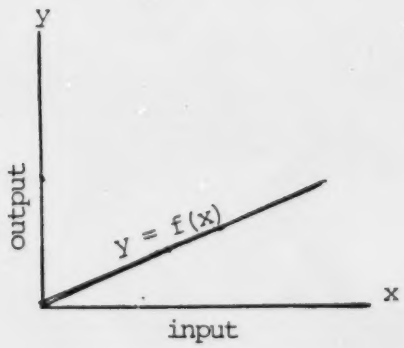


we assume for the moment that we know precisely the relationship between input and output under conditions of perfect efficiency. That is,

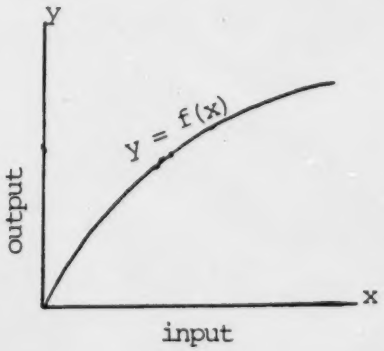
$$\text{output} = f(\text{input})$$

$$y = f(x)$$

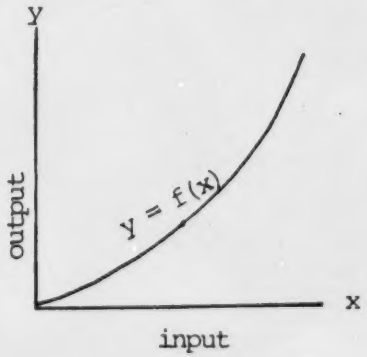
We call this function the efficient production function. A graph of this function would enable us to determine the exact amount of output that could be produced from any given amount of input by a perfectly efficient unit, industry, or institution. The concepts of constant, increasing, and decreasing returns to scale are illustrated by the following graphs of efficient production functions.



constant

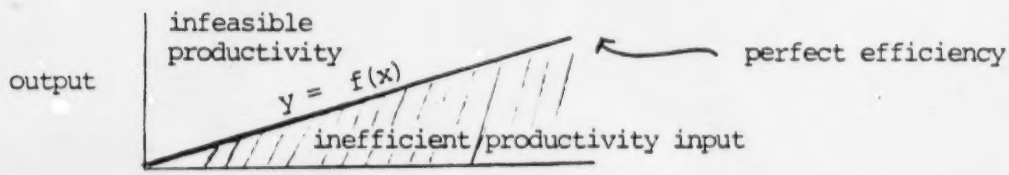


decreasing returns to scale

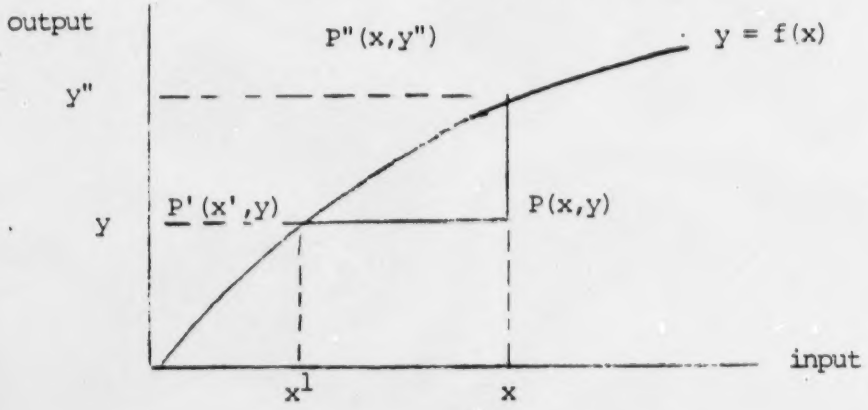


increasing

We should note that an efficient production function divides the space into two sets of points. The set of points below the curve represent pairs of inputs and outputs that are possible but not optimal. That is, they are characteristic of an inefficient unit. The points above the curve are combinations of inputs and outputs that are beyond the ideal limits of perfect efficiency and are thus impossible; such points are called <u>infeasible.</u>



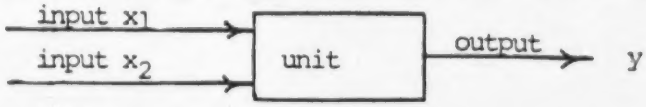
While we are looking at this very simple example it may be instructive to see why any unit below the curve is considered inefficient.

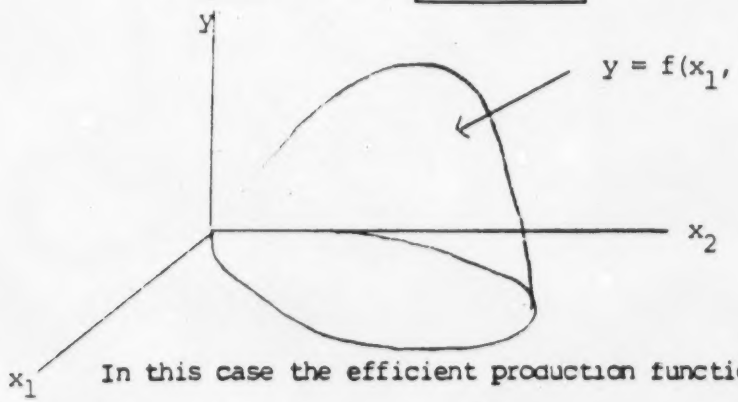


Consider institution P which uses X amount of input to produce Y amount of output. Clearly, it is possible to find a point P'(X',Y) which produces the same output Y but uses less inputs X'.

It is also possible to find another efficient institution P"(X,Y") that uses the same inputs (X) as P but produces a larger output (Y"). Thus, the inefficiency of P is related to how far the point is from the efficient production curve.

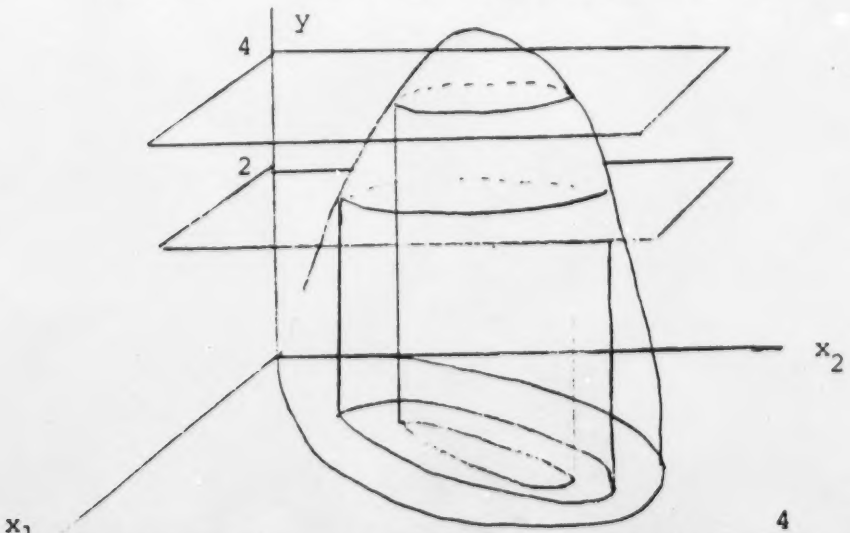
We can only go a short step from this simplest situation before the situation becomes too complex to pictorially represent. The most complex production function that we can draw is one consisting of one output and two inputs.





In this case the efficient production function graphs into a surface in three-dimensional space rather than a curve in two-dimensional space as before. Such drawings depend upon perspective and accurate scales and are thus awkward to work with. For this reason, economists have used other, more abstract ways to represent these surfaces.

One technique is to draw a contour map of the surface. If planes are passed parallel to the base at different elevations, they will slice the surface in curves. These curves can then be projected down on the base plane and labelled by their elevation. One soon develops a facility for visualizing the original surface by looking only at such contour maps.



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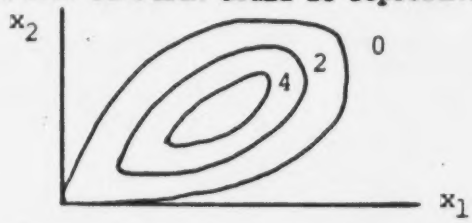
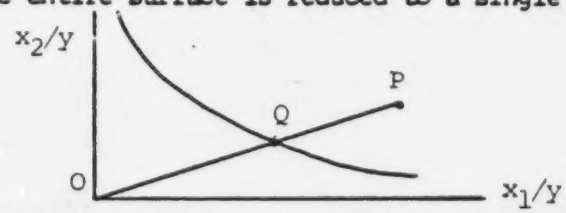
The above production function could be represented by:



An even handier device is to divide each input by the output, getting X_1/Y, X_2/Y. If the points $(X_1/Y, X_2/Y)$ are plotted (assuming constant returns to scale), the entire surface is reduced to a single curve.



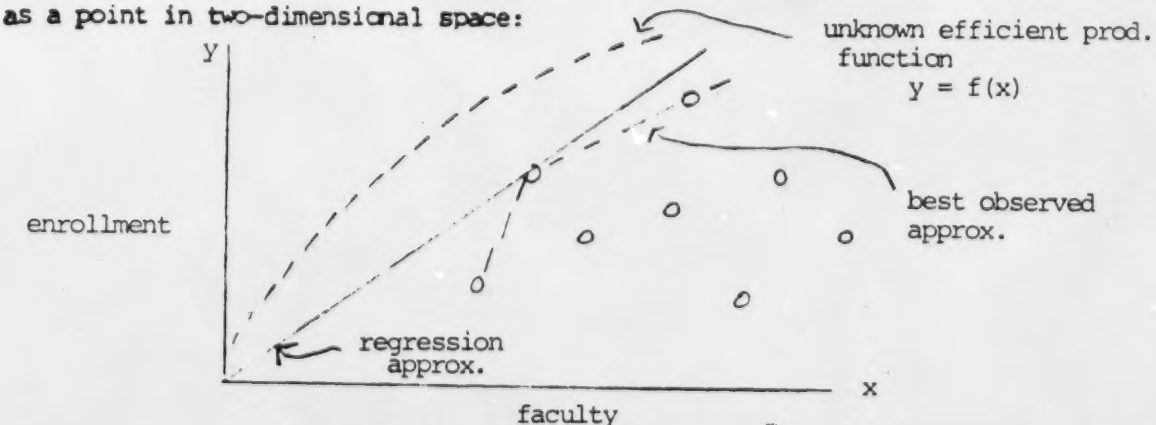
Much can be done with such a plot. If P represents a particular institution, then it is clearly inefficient. A measure of its inefficiency is how far it is off the production curve. It can be shown that Q represents the location P would occupy if it were totally efficient. Also the ratio $\frac{OQ}{OP}$ is a reasonable measure of P's efficiency.

(Note: $\frac{OQ}{OP} \longrightarrow 1$ as $P \longrightarrow Q$)

Without going into the economical theory any further, let us turn to the problem of finding such efficient production functions.

Farrell's Method

Suppose we assume that we have data on many institutions. Let the output be "enrollments" and the input be "faculty". Each institution can be plotted as a point in two-dimensional space:



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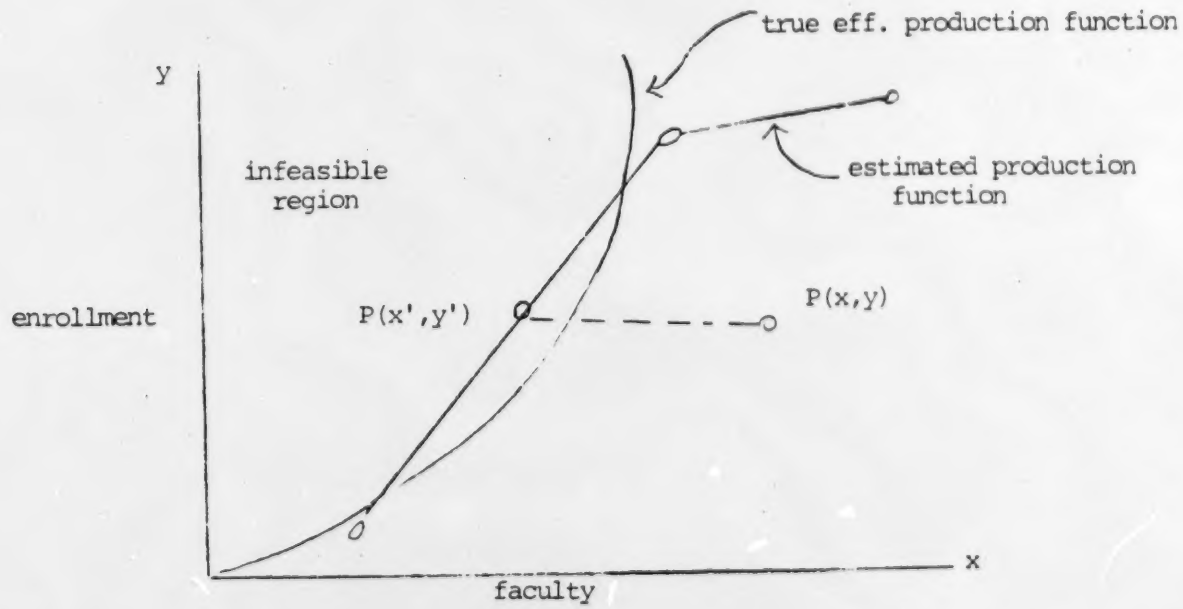
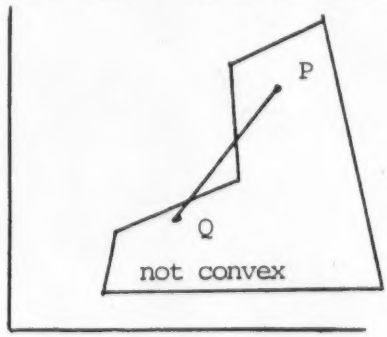
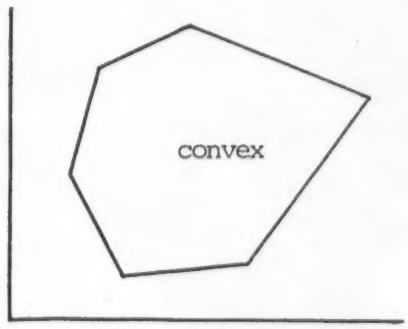
How can we estimate the unknown efficient production function for these institutions? One crude method is to use regression methods to fit a line to these points and constrain the residuals to be positive, which forces all the points to lay below the line. The economist Farrell reasoned that in the absence of information the best estimate of the production function is that represented by the most efficient units observed. Thus, he would draw the segmented curve shown in red as the most reasonable, conservative estimate.

At this point the term "convexity" must be injected in order to proceed. Farrell's method consists of connecting all the "extreme" points with line segments. This clearly results in a boundary or "frontier" line which represents the best observed approximation to the unknown production function curve. We now wish to claim that any point (X,Y) on such a frontier line can be realized, i.e., it is possible for some unit (institution) to achieve that enrollment (Y) for that particular faculty count (X). If care is not taken, this may not be possible, as illustrated below:

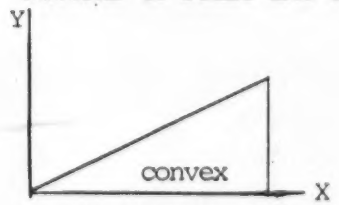


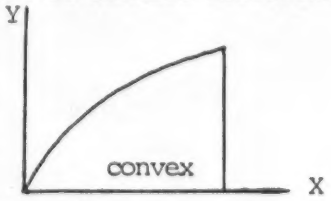
Point P is not efficient and if we use the estimated production function, we would claim that P should be able to increase its efficiency until it moves to position P′. But P′ is a point in the infeasible region of the true production function and so it is impossible for any institution to support an enrollment of Y′ using only X′ number of faculty. For this reason, it is necessary to insist that the feasible region of a given production function be convex, i.e., of such a shape that every two points in the feasible region can be connected by a line segment laying entirely within the region.

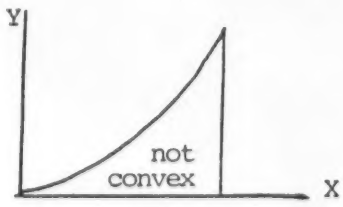




The production functions for processes that have constant or decreasing returns to scale are always convex. This is not the case for those with increasing returns to scale and these must be handled differently.

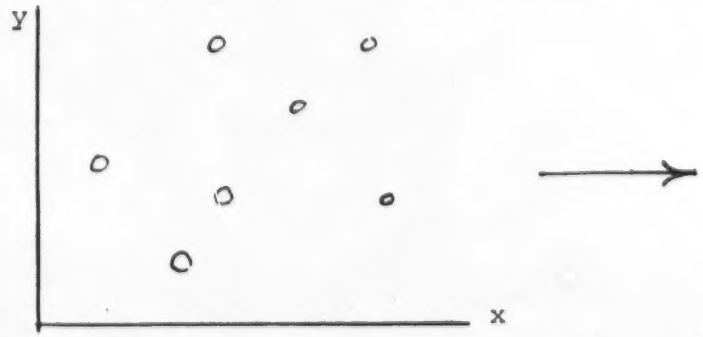


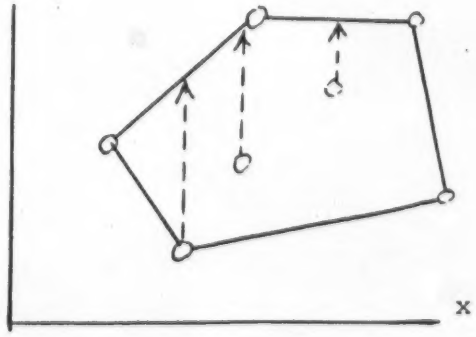




In what follows, we assume convexity.

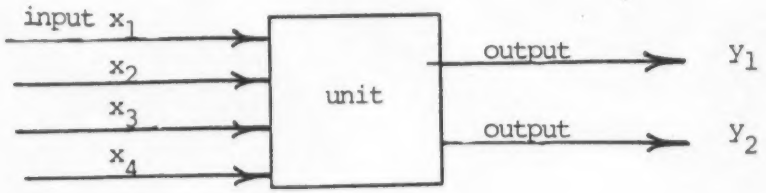
Farrell considered the plot of all production units and proposed that they be inclosed in the smallest possible convex shape.





He then measured the efficiency of each unit (point) by a ratio type measure of how far it was from the given point to the frontier of the convex region in the direction of some selected output.

From this point on, the problem became one of technique. In 1971, Boles showed how the tremendous number of computer calculations involved in Farrell's method could be greatly reduced by formulating the problem in terms of a linear programming problem and using any one of many readily available software packages to study the most general problem—processes involving both multiple inputs and multiple outputs (i.e., joint products). These are processes with the form:



or $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = f\ (x_1, x_2, x_3, x_4)$

Thus, an educational institution could be considered as a process using inputs such as senior faculty, junior faculty, equipment, classroom space and producing outputs such as graduates and research papers. Farrell's method would consist of representing each institution as a point with coordinates $(Y_1, Y_2, X_1, X_2, X_3, X_4)$. If all the institutions are plotted in six dimensional space and then joined by line segments to enclose smallest possible convex region, we get a six dimensional shape called a polytope. Even though we can't see it or draw it, we can tell how many "sides" (called facets) it has and measure how far every point (institution) is from any given

side. If we consider the surface of the polytope to be the frontier productivity function, then the distance to the surface from each point measures the production efficiency of the points.

Carlson's Work

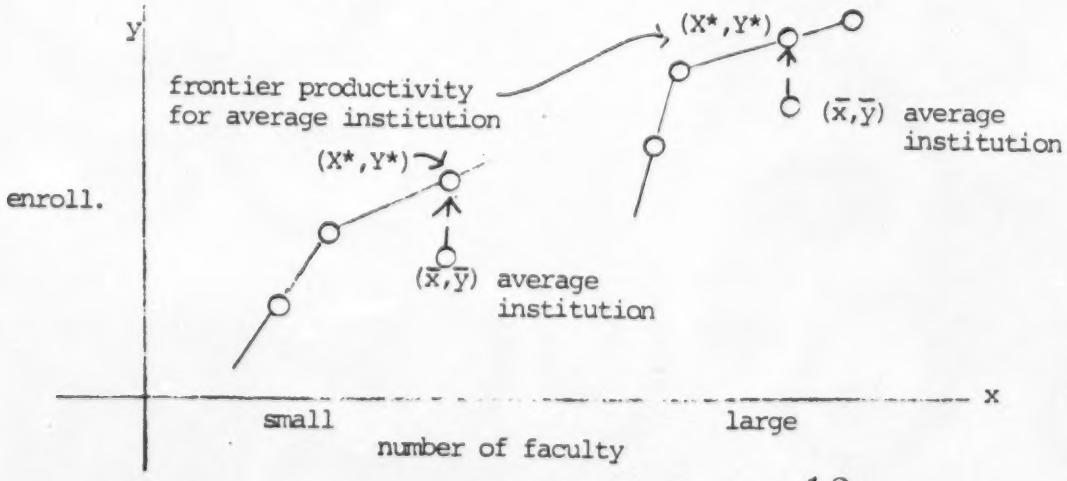
D. Carlson used Farrell's method as refined by Boles to analyze the efficiency of 1,330 higher education institutions, ranging from small, non-selective private colleges to large public institutions. His dissertation (which later became part of the Ford Foundation Program at Berkeley) is interesting for several reasons:

(1) he showed that Farrell's method worked—i.e., it produced results that were intuitively reasonable;

(2) some of the results were surprising and all were very much different from those obtained from conventional methods of averaging various ratios; and

(3) he demonstrated various insights that could be obtained by different formulations of the linear programming problem.

One issue explored by Carlson was the comparison of conventional "efficiency" ratios of students/faculty and cost/student with similar ratios calculated using the efficient production function estimated by Farrell's method. His model was many dimensional but his general methodology can be explained using a simple two dimensional example as follows:



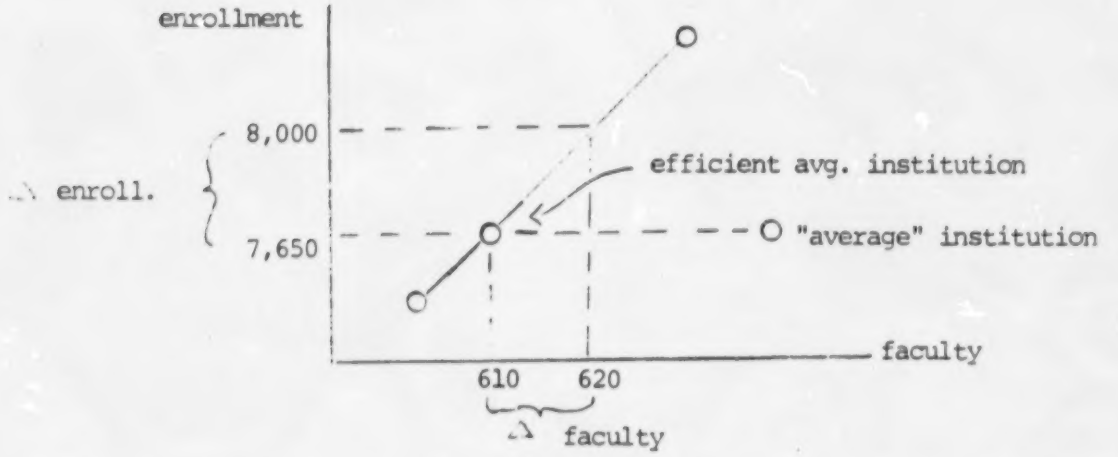
Given a set of institutions represented by their enrollments and faculty size, we can partition them into two sets—large and small institutions—so as to allow for increasing returns to scale. For each group we could determine the average enrollment ($\bar{Y}$) and faculty size ($\bar{X}$) and this plot could be considered as the "average" (typical) institution ($\bar{X}$, $\bar{Y}$). For such an "average" institution, we can calculate the conventional student/faculty ratio $\bar{Y}/\bar{X}$ which proports to measure efficiency.

The second step would consist of determining what the "average" institution should be able to produce if it were totally efficient. This is achieved by holding the resources constant (faculty) and maximizing the output (enrollment). This, graphically, is equivalent to moving the average institution up to the efficient production frontier (X^*, Y^*) and calculating the student/faculty ratio Y^*/X^* . Carlson displays several comparisons of such "average" and "frontier" ratios.

A Comparison of Average and Frontier Efficiency Measures

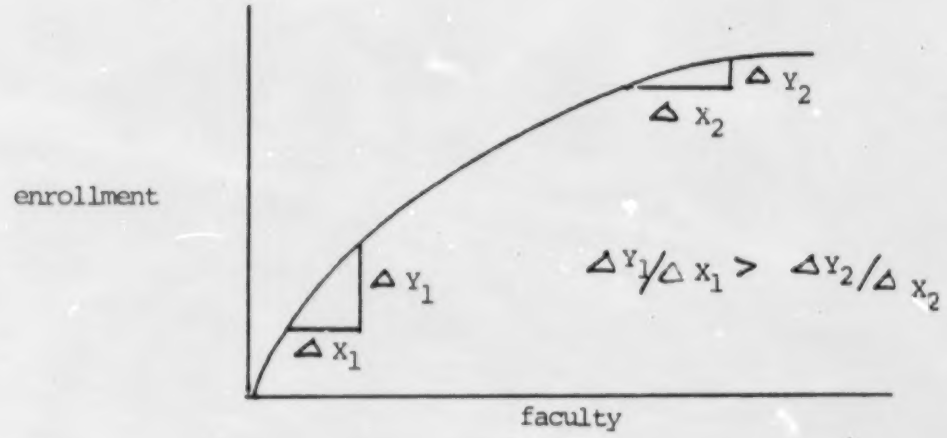
Type Institution	Student/Faculty		Cost/Student	
	Avg.	Frontier	Avg.	Frontier
Selective Liberal Arts College	13.8	18.7	$1,371	$ 768
Public Comprehensive University	32.1	23.5	$ 883	$ 514

Carlson also does some interesting analyses of marginal productivity. Again, the method is best shown with a simple two dimensional example:



Starting with the hypothetical 'average' institution, he holds the output constant and finds the minimal faculty needed under efficient operation (done by moving to the frontier). He then changes the enrollment and sees what the corresponding minimal faculty should be for a frontier institution. The calculation $\frac{\Delta \text{enroll}}{\Delta \text{faculty}}$ is the marginal productivity relationship for an institution with the characteristics of the average institution. By repeating this calculation with increasing Δ(enrollments) for the institutions grouped by type, he found that:

(1) the marginal productivity for $\frac{\Delta \text{ enroll}}{\Delta \text{ faculty}}$, and $\frac{\Delta \text{ enroll}}{\Delta \text{ classroom space}}$ decreased as the size of the institution increased (for all types of of institutions). This demonstrated the expected decreasing returns to scale.



(2) For the institutional categories, the marginal productivities for the private institutions were higher than for their public counterparts. A sample table from Carlson's 1972 study appears below.

Marginal Relationships - Public Universities

Variable	Observation 1		2		3		4		5
Full-time Undergraduates (UGFT)	3,820		7,650		9,539		11,480		
Minimum Senior Faculty (SFAC)	594		595		651		788		
ΔUGFT/ΔSFAC		3,830.0		33.8		14.2			
Part-time Undergraduates (UGPT)	835		1,041		1,250		2,080		3,120
Minimum Senior Faculty (SFAC)	651		651		651		651		654
ΔUGPT/ΔSFAC		-		-		-		347.0	
Graduate Students (GRAD)	1,080		2,160		2,697		3,230		
Minimum Senior Faculty (SFAC)	617		618		651		704		
ΔGRAD/ΔSFAC		1,080.0		16.3		10.0			
Specialized Enrollment (OTHE)	0		400		498		600		
Minimum Senior Faculty (SFAC)	602		627		651		684		
ΔOTHE/ΔSFAC		16.0		4.1		3.1			
Full-time Undergraduates (UGFT)	7,650		9,539		11,480				
Minimum Classroom Space (CLSP)	97,060		118,454		151,792				
ΔUGFT/ΔCLSP		.088		.058					

Carlson also uses his model to demonstrate how a collection of institutions can be ranked according to their production efficiency (and also their cost efficiency). In one study he used a "simplified model to minimize each institution's faculty size with the other institutional variables held constant. Each institution's measure of efficiency was experssed in terms of how its current faculty size compared to the "efficient" faculty size.

Thus, in the table below (Carlson, 1972) we see that institution 114 compares identically, i.e., it must be on the production frontier, whereas institition 101 has 30.6% more faculty than what an efficient institution of that size and type should have.

Measures of Production Efficiency—Private Highly Selective Liberal Arts Colleges

	Simplified[a]		*Expanded*[b]	
Institution	*Measure*	*Rank*	*Measure*	*Rank*
114	1.000	1	1.000	1
117	1.221	2	1.210	8
120	1.298	3	1.256	9
101	1.306	4	1.000	1
123	1.355	5	1.000	1
112	1.458	6	1.081	6
110	1.511	7	1.000	1
103	1.543	8	1.100	7
105	1.630	9	1.484	10
109	2.654	10	1.000	1

[a] Minimum faculty with full-time-equivalent enrollment and size of the institution constrained.

[b] Minimum faculty with full-time undergraduates, part-time undergraduates, graduate students, teaching assistants, classroom space, laboratory space, research commitment, number of fields, acceptance rate, and size of the institution constrained.

Carlson then increases the complexity of the model by including more institutional variables. Intuition tells us that this second model should result in more institutions being classified as "efficient" because it takes into consideration a broader mix of characteristics—a mix which was collapsed into a single variable by the simplified model. Geometrically, this result is also expected since the additional variables add more dimensions to the space and the resulting polytope has many more facets and thus more institutions on its surface.

The "expanded" model data in the table shows that in addition to institution 114, institutions 101, 123, 110, and 109 are now "efficient". This demonstrates that the efficiency of an institution is greatly dependent upon what variables and characteristics we are willing to take into account.

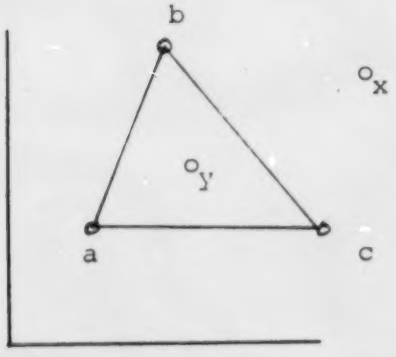
Gray's and Lindsay's Work

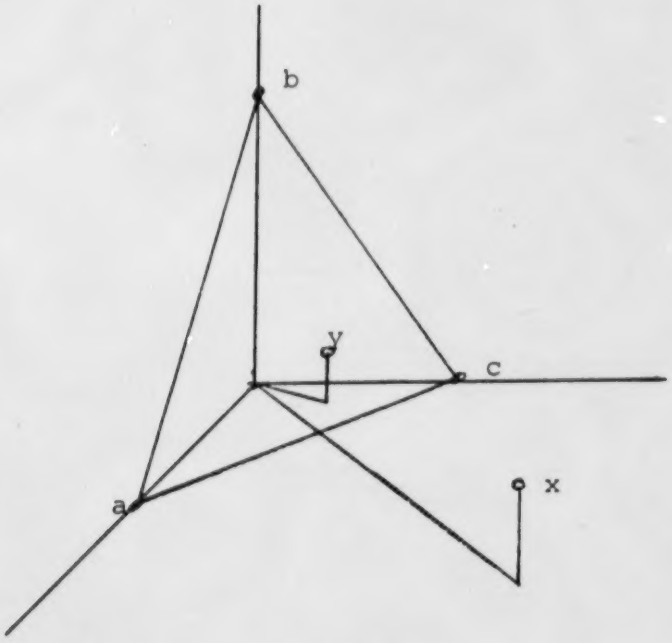
As Boles and Carlson demonstrated, a researcher can use Farrell's techniques by using a standard linear programming computer software package to construct an n-dimensional polytope and measure distance from interior points to the surface; however, the freedom from this programming chore is bought with the labor of formulating a different linear programming model for every different aspect of the study. Thus, it was only a matter of time until someone decided that it may be worthwhile to invest the effort to build a model that would thence forth relieve the researcher of any further programming or problem formulating labors. This was done by R.G. Gray, who in 1977-79 developed such a computer model both as a thesis project and as a project for NCHEMS. This program, which did not use a linear programming approach, was capable of building a polytope in five (5) dimensions (enough variable to handle a large assortment of studies in higher education). Gray used his model to analyze a set of chemistry departments in universities. (His program is currently

available from NCHEMS). In 1979, A.W. Lindsay extended the methodology of Gray's program to handle twelve (12) dimensions (variables) and a data set of 300 points with the polytope limitations of 1,800 facets and the condition that not more than 90 facets are incident at any one vertex.

The difficulties encountered in constructing a n-dimensional polytope are not apparent to one working with pencil and paper on a two (2) or three (3) dimensional example.

The major difficulty is in deciding whether or not a given point should be considered a frontier (surface) point or not. In the two (2) or three (3) dimensional examples, it is obvious that 'x' should be added as a boundary point and that 'y' should not.





The Polytope construction proceeds as follows:

(1) Given a K-dimensional space, select 'K' points that are linearly independent (i.e., no pt., considered as a vector, can be constructed from any combination of the other (K-1) pts. (vectors). This is always possible to do.

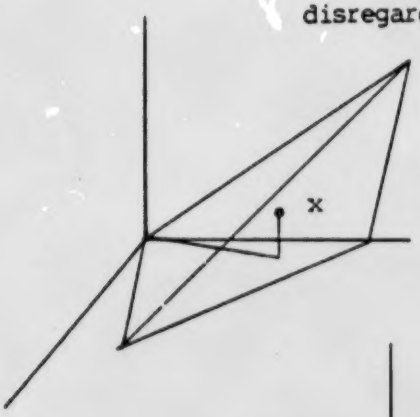
(2) From the 'K' selected pts., construct a K-dimensional polytope.

(3) Consider each of the remaining pts. one at a time and treat as follows:

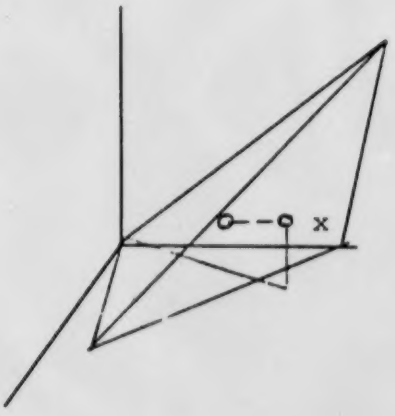
(i) If the pt. lays on an existing facet of the polytope - displace it slightly towards the center and regard it as an interior pt.

(ii) If the pt. lays on the same side of the polytope as the centroid, regard it as an interior point.

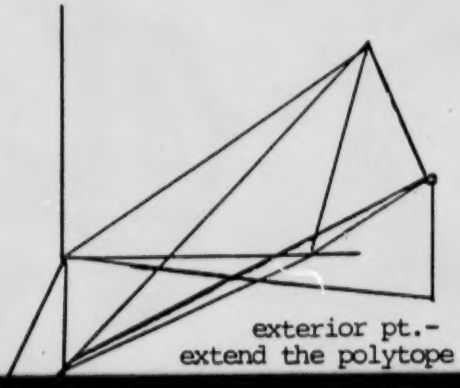
(iii) If the pt. lays on the outside of the polytope, define new facets by connecting every polytope vertex to the new pt. and then disregard any former vertices that are now interior points.



interior pt. - disregard



facet pt. - displace toward the interior



exterior pt.- extend the polytope

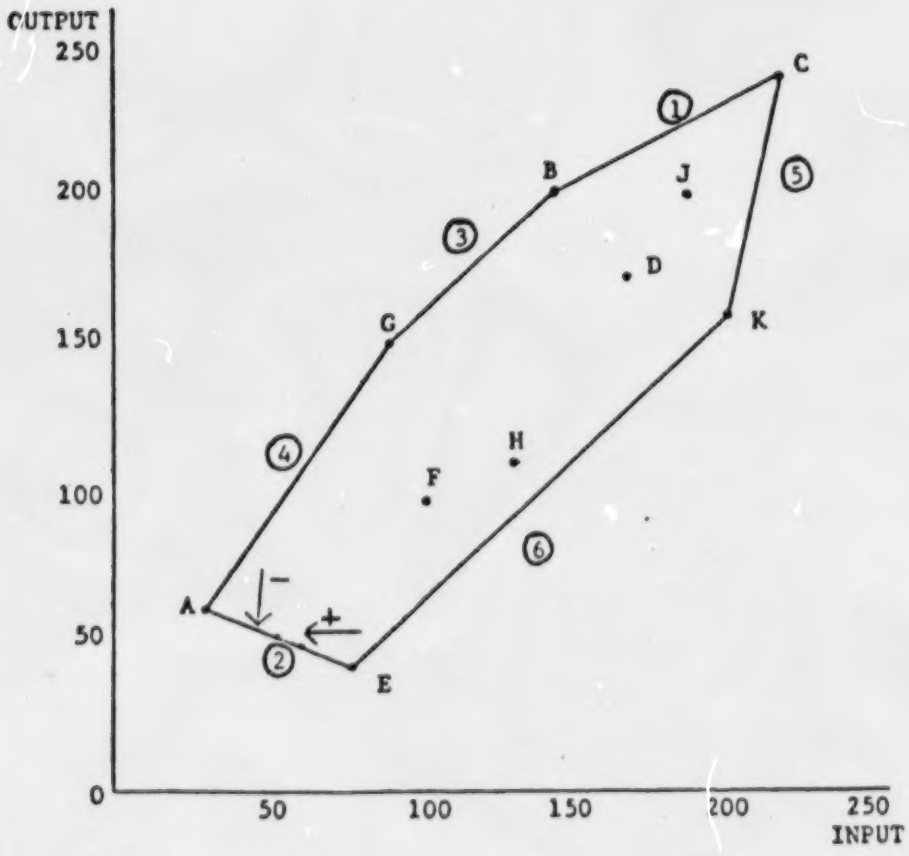
Lindsay used his program to analyze the collection of all Australian universities across four and seven variables and placed into various efficiency categories.

In order to identify those facets that represent the efficient production function, Lindsay designed his program to assign a Dimensional Direction Score (DDS) to each facet. The following two-dimensional illustration demonstrates this method.

HYPOTHETICAL DATA SET USED IN ILLUSTRATING THE APPLICATION OF THE CONVEX POLYTOPE MODEL
(Lindsay, 1979)

Institution	Input	Output
A	30	60
B	140	200
C	220	240
D	170	170
E	80	40
F	100	100
G	90	150
H	130	110
J	180	210
K	200	160

Given the above data, the program would construct the two dimensional polytope as explained before. (Lindsay, 1979)



The computer then classifies each facet in terms of the extent to which it is extreme in the desired direction - using '+' and '-' to denote whether the facet is extreme in the desired or undesired direction respectively. In our example, the output '+' direction is 'up' and the input '+' is 'left'. The red and blue arrows indicate how facet 2 is classified as a (+,-) facet. Each

facet is then given a DDS score by treating each '-' as 0 and each '+' as 100 ÷ dimension (= $\frac{100}{2}$ = 50, in our example) and summing them all up. Thus, the score for facet (2) is 0 + 50 = 50. For facet (4) which is (+,+), the DDS = 50 + 50 = 100.

Facet	Dimensional Dir. Code (input, output)	Dimensional Dir. Score
1	+ +	100
2	- +	50
3	+ +	100
4	+ +	100
5	- -	0
6	- -	0

Thus, we see that facets (1), (3), and (4) represent the approximation to the efficient productions function. The computer then calculates and displays the distance from each point to the efficient facets in terms of each dimension successively with the other dimensions held constant. This distance is recorded in terms of percent of each point's original value on that coordinate. Thus, we get:

SINGLE DISPLACEMENT MEASURE

FOR THE TWO-DIMENSIONAL ILLUSTRATION (Lindsay, 1979)

	Displacements (% of existing value) to move to surface. Facet reached in parens.	
	DIMENSION	
Institution	Input	Output
A	0	0
B	0	0
C	0	0
D	-35.3 (3)	+26.5 (1)
E	0	+237.5 (4)
F	-43.3 (4)	+60.0 (3)
G	0	0
H	-51.3 (4)	+72.7 (3)
J	-11.1 (1)	+4.8 (1)
K	-50.0 (3)	+43.7 (1)

From this table we see that points (institutions) D,E, F,H,J,K are not 'efficient' and should be able to either increase their output or decrease their inputs with no adverse effects on the other dimensions.

For a two dimensional example, such coding and classifying of facets and points may seem superflous but once one begins to work with five and six dimensional models with hundreds of facets and points, some such procedure is necessary to bring order to chaos.

Lindsay also defined several other measures of distances to the surface (MINIMAX and MINISUM), which would be familiar to people working in operations research; however, these appeared to be less than successful in the context of such polytopes because of interpretation difficulties.

Conclusion

As stated at the outset, this presentation was intended as but an introduction to a potentially useful technique. Clearly there are many theoretical and technical intricacies that have been left unexplained with the expectation that interested researchers can gain an understanding of them by reading the referenced articles or by working on specific studies using these techniques.

Farrell's method seems most appropriate for measuring the efficiency of constituent units within a system. Such systems as universities and colleges within a state higher education system, various campuses comprising a multi-campus university, or a collection of departments of the same discipline drawn from a group of peer institutions, all would appear to be equally amenable to analysis by this technique. If but one or two researchers undertake such a study, the methodology will at least be kept active and perhaps even extended. Until enough evidence in the form of attempted studies or successful outcomes is on hand, we are not able to evaluate the value of this methodology, and it may be prematurely consigned to the graveyard of forgotten lore.

Computer Programs

Linear Programming Approach

Carlson (1975, 1976) provides several L.P. formulations to Farrell's method. Presumably Boles (1970) also presents some L.P. models.

Polytope Approach

NCHEMS has a FORTRAN program developed by Gray and Weldon, which can be obtained together with control cards, a test run, and a copy of A Convex Hull Approach to the Analysis of Social Productivity, for $75, although the program was not intended and is not maintained as a portable software product.

NCHEMS would also provide at cost the following papers:

Robert Gray, H. Kent Weldon, Leonard C. Romney, "A Conceptual Framework of Departmental Performance," 1979.

Robert Gray and Kent Weldon, "An Experiment with Convex Product Functions," 1978.

Robert Gray, "Some Models and Solution Techniques for Analyzing Student Flow Processes," 1974.

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